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On the Role of Average Values in Solving Geometric Extreme Problems

J. Husanov, N. Shamsiddinov
Tashkent state technical university named after I.Karimov, university street №2, Tashkent siti, Republic of Uzbekistan
Nshamsiddinov1974@gmail.com

Abstract: In the article, in addition to studying the properties of functions when finding extreme values, methods for applying the average values of expressions in functions and relations between them were studied.

Keywords: Geometric, Mathematical, Extreme.

The technological progress of mankind is always subordinated to the task of not just fulfilling the set goal, but fulfilling it with the least cost and the greatest result, that is, rational, optimal fulfillment of the task. The current stage of development is no exception. It is not enough to propose a particular process or algorithm for implementation, it is more important to justify its expediency and effectiveness in comparison with existing processes. Mathematical analysis is one of the tools of such a rigorous justification.

The basics of the skills of finding the optimal solution are laid already in the school course of mathematics. In algebra textbooks for the 9th grade of a secondary school, there are often entertaining tasks for finding extreme (largest or smallest) values of a function [1]. It should be emphasized that students of this already significant age are just beginning to get acquainted with them. With a full set of tools, they are found only in the final grades when studying the basics of differential calculus. In the textbook for the 11th grade [2], the derivative is introduced in the 1st half of the year. By this time, the motivation to study mathematics is irretrievably lost, as graduates face important events, such as preparation for state final exams at school and university entrance exams.

Meanwhile, solving extreme problems can become an effective tool for the formation and development of creative mathematical skills of students. Of course, schoolchildren should be "armed" with mathematical tools to solve optimization problems of various levels without premature use of the derivative. It is in the 9th grade that the concept of a quadratic function is introduced. The use of its extreme properties is often quite sufficient to solve optimization problems offered in the course of differential calculus.

Let us recall the main facts related to the use of the quadratic function as a tool for solving extremum problems [3].

The function is called quadratic if it can be defined by the formula $y = ax^2 + bx + c$, where x is an independent variable, a, b and c are some real numbers (coefficients), and $a \neq 0$. The domain of the quadratic function definition is the entire numeric axis: $D = (-\infty; +\infty)$; E − the set of values is a ray: either, $E = (-\infty; y_{\max}]$ or $E = [y_{\min}; +\infty)$. It is convenient to study the quadratic function in the form containing the selected full square: $y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$.

There are two possible cases:

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1) If $a > 0$, this term $a\left(x+\dfrac{b}{2a}\right)^2$ is positive everywhere and $x_0 = -\dfrac{b}{2a}$ only vanishes when. Therefore, the function y has the smallest value $y_{\min} = c - \dfrac{b^2}{4a}$ and does not have the largest, and, $x_0 = -\dfrac{b}{2a}$ is the only point of the minimum.

2) If $a < 0$, then already the largest value of the function $y_{\max} = c - \dfrac{b^2}{4a}$ is reached at $x_0 = -\dfrac{b}{2a}$, and the smallest value does not exist. Then $x_0 = -\dfrac{b}{2a}$ – the only point of the maximum.

The above can be justified graphically. The graph of a quadratic function is a quadratic parabola with branches pointing up or down, depending on the sign of the higher coefficient. Obviously, the smallest (at $a > 0$) or largest (at $a < 0$) value of the function will be the values at the top of the parabola.

In the curriculum, a special place is allocated for finding the largest and smallest values of different expressions. In particular, before the introduction of the concept of derivative, the study of the extreme value begins and it uses the average values of expressions and the relations between them. But in the current curricula and manuals, information about them is not enough for detailed study.

Generically, the smallest and largest values are called the extreme values of the function. The above can be expressed in the form of a theorem, which is the main one for solving extreme problems using the properties of a quadratic function without using a derivative.

From a high school course, we know that there is a relation between the arithmetic mean and the geometric mean of $\sqrt{ab} \le \dfrac{a+b}{2}$ two positive numbers a and b and they will be equal only if $a = b$. The algebraic proof of this inequality is simple:

$$0 \le (a-b)^2 \Rightarrow 4ab \le (a+b)^2 \Rightarrow ab \le \left(\frac{a+b}{2}\right)^2 \Rightarrow \sqrt{ab} \le \frac{a+b}{2}.$$

From high school, we know that in a right triangle, the height drawn from a right angle is equal to the average geometric projection of the legs on the hypotenuse. Let's take advantage of this

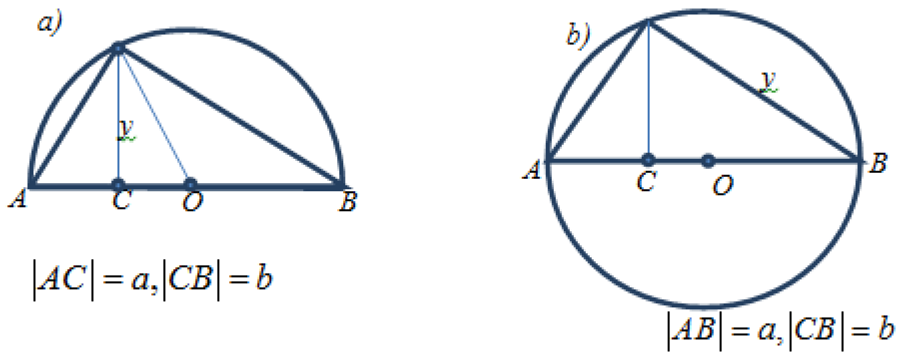

Figure 1.

The geometric mean of two numbers is also called the proportional mean because in $y = \sqrt{ab}$ the numbers a, y and b make up a geometric progression, and a in $\dfrac{a}{y} = \dfrac{y}{b}$ y is the average member of the proportion.

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To divide some value of a in the "golden section", you need to divide it into such parts *y* and *a-y*, in which the value of y should be equal to the geometric mean of *a* and *a-y*:

$$\frac{a}{y}=\frac{y}{a-y}$$

Hence: $y=\frac{\sqrt{5}-1}{2}a\approx 0{,}618a$

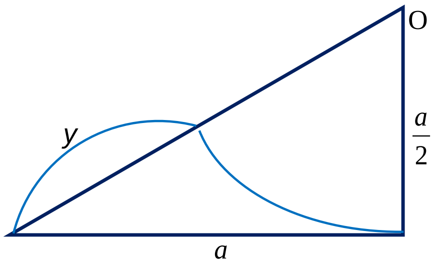

Figure 2.

When studying educational materials, it is useful to use the following double inequalities:

$$\frac{2ab}{a+b}\le\sqrt{ab}\le\frac{a+b}{2}\le\sqrt{\frac{a^2+b^2}{2}}\quad(1)$$

$\sqrt{ab}\le\frac{a+b}{2}$ was proved at the beginning.

The proof of the rest of the for,

1. $\left(\frac{a+b}{2}\right)^2-\frac{a^2+b^2}{2}=-\frac{1}{4}(a-b)^2\le 0$, means $\frac{a+b}{2}\le\sqrt{\frac{a^2+b^2}{2}}$

2. $\left(\frac{2ab}{a+b}\right)^2-ab=-ab\left(\frac{a-b}{a+b}\right)^2\le 0$ so, $\frac{2ab}{a+b}\le\sqrt{ab}$.

The student can check that equality will hold if $a=b$.

Let a straight line segment be drawn in a trapezoid whose bases are equal to a and b. If the segment:

1) Divides the area of the trapezoid into two equal parts , then the length of the segment is equal to the average of the quadratic values of a and b (3-figure $EF=\sqrt{\frac{a^2+b^2}{2}}$);

2) Divides the trapezoid into two similar ones, then y gives the geometric mean a and b (4-figure, $RS=\sqrt{ab}$);

3) Passes through the intersection point of the diagonals, then y is equal to the harmonic mean a and b (5-figure, $PT=\frac{2ab}{a+b}$).

Proofs. 1) Let the segment EF parallel to the bases in the trapezoid ABCD divide the area of the trapezoid into two equal parts $S_{DEFC}=S_{EABF}$ and $DL\|CB\|EK$, $h_1\perp EF$, $h_2\perp AB$. By condition $\frac{b+EF}{2}\cdot h_1=\frac{EF+a}{2}\cdot h_2$ or $\frac{h_1}{h_2}=\frac{EF+a}{b+EF}$. But by $\Delta EDL \backsim \Delta AEK$ $\frac{h_1}{h_2}=\frac{EL}{AK}$, then

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$EL = EF - b$, $AK = a - EF$. then $\frac{EF+a}{b+EF} = \frac{EF-b}{a-EF}$, from here $EF = \sqrt{\frac{a^2+b^2}{2}}$.

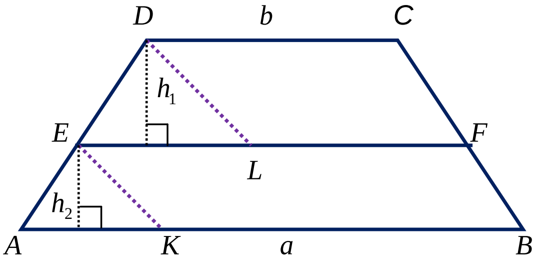

Figure 3

2) Let the segment RS divide $ABCD$ into two similar $ARSD$ and $RDCS$ trapezoids (4-figure), because $\frac{b}{RS} = \frac{RS}{a}$, $RS = \sqrt{ab}$;

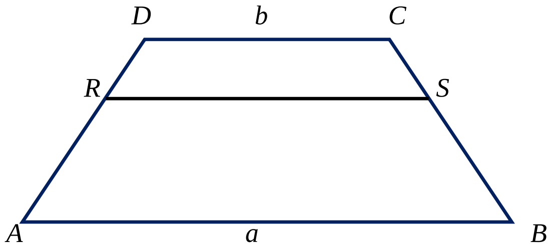

Figure 4

3) Let the segment PT parallel to the bases pass through the intersection point O of the diagonals BD and AC (5-Figure). By the property of similar triangles,

$\frac{AP}{PD} = \frac{AO}{OC} = \frac{a}{b}$.

$\frac{PO}{a} = \frac{DO}{DB} = \frac{CO}{CA} = \frac{OT}{a}$, hence $PO = OT$ and

$\frac{PO}{a} + \frac{OT}{b} = \frac{DO}{DB} + \frac{BO}{DB} = \frac{DO+BO}{DB} = \frac{DB}{DB} = 1$. Therefore,

$$PT = 2 \cdot PO = 2 \cdot \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a+b};$$

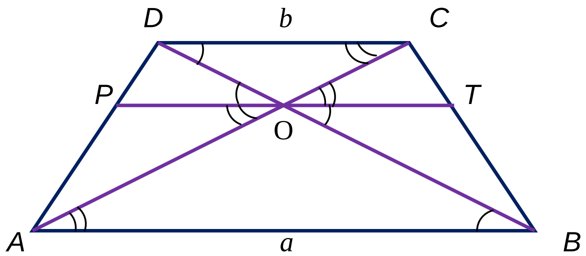

Figure 5.

3. in Figure 6, let be AV - the tangent , $AB = a$, $AC = b$, $b < a$, radius is $OC = \frac{a-b}{2} = OE = OV$. Then:

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1) $AO = AC + CO = b + \frac{a-b}{2} = \frac{a+b}{2}$ - arithmetic mean;

2) For tangents AV and AU, $AV^2 = ab$, $AV = \sqrt{ab}$ is the geometric mean;

3) $EO \perp CB, AE^2 = AO^2 + OE^2 \Rightarrow AE = \sqrt{\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2} = \sqrt{\frac{a^2+b^2}{2}}$ is the average quadratic.

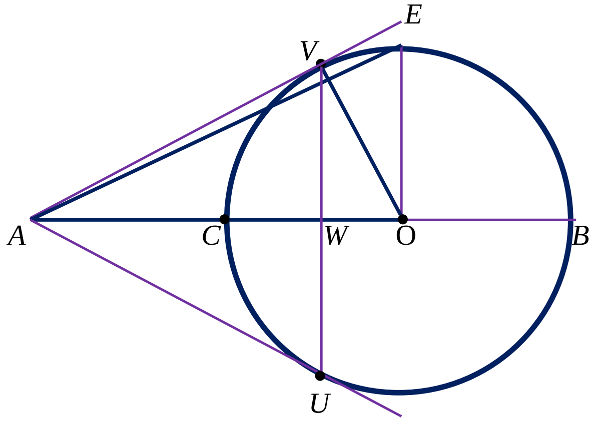

Figure 6

In general, problems of finding the average value are quite common in high school.

It is known that the course of mathematics of higher classes gives some inequalities $\sqrt{ab} \le \frac{a+b}{2}$, which we considered at the beginning, in the form $\sqrt[n]{a_1 a_2 ... a_n} \le \frac{a_1 + a_2 + ... + a_n}{n}$ where $a_i > 0$ (Cauchy inequality). For positive and negative numbers $a_1,..., a_n$, the inequality $\left|\frac{a_1 + ... + a_n}{n}\right| \le \sqrt{\frac{a_1^2 + ... + a_n^2}{n}}$. These inequalities will be equal if $a_1 = a_2 = ... = a_n$.

Many problems are solved using the inequalities discussed above.

Most often when finding the largest inequality problems $\sqrt{ab} \le \frac{a+b}{2}$. When $a = b$ the expression $\frac{a+b}{2}$ gives the smallest value, which is equal to $\sqrt{ab}$.

This statement remains valid for the expression $\frac{2ab}{a+b}, \sqrt{\frac{a^2+b^2}{2}}$ and for any n positive numbers that make up the general form of the expression.

Here are some typical tasks.

1. Find the smallest value of the function $y = x + \frac{A}{x}$, $x > 0$, $A > 0$.

Solution: by inequality $\frac{a+b}{2} \ge \sqrt{ab}$ ($a > 0$, $b > 0$):

$$\frac{x + \frac{A}{x}}{2} \ge \sqrt{x \cdot \frac{A}{x}} = \sqrt{A},$$

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Hence it turns out that in $x = \frac{A}{x}$, or $x = \sqrt{A}$ then $y_{min} = 2\sqrt{A}$.

2. At what values of x, the function $y = \sqrt{\sin^2 x + A\cos^2 x} + \sqrt{\cos^2 x + A\sin^2 x}$

($A > 0$) takes the largest value?

Solution: if at some value, the function takes the largest value, then, according to the condition of the problem, it also takes its largest value. We use this property:

$$y^2 = \left(\sin^2 x + \cos^2 x\right)(1 + A) + 2\sqrt{\left(\sin^2 x + A\cos^2 x\right)\left(\cos^2 x + A\sin^2 x\right)}$$

or $y^2 = 1 + A + 2\sqrt{\left(\sin^2 x + A\cos^2 x\right)\left(\cos^2 x + A\sin^2 x\right)}$.

The value of the variable under the root, - the constant largest value coincides with the largest expression under the root. By the inequality of the geometric mean and the arithmetic mean , this value takes the form:

$$\sqrt{\left(\sin^2 x + A\cos^2 x\right)\left(\cos^2 x + A\sin^2 x\right)} \le \frac{\left(\sin^2 x + A\cos^2 x\right) + \left(\cos^2 x + A\sin^2 x\right)}{2}$$

or

$$2\sqrt{\left(\sin^2 x + A\cos^2 x\right)\left(\cos^2 x + A\sin^2 x\right)} \le 1 + A.$$

Means, $y^2 \le 2(1 + A)$.

If $A \neq 1$ the expression becomes equal at $\sin^2 x + A\cos^2 x = \cos^2 x + A\sin^2 x$ or $A\cos 2x = \cos 2x$ or $\cos 2x = 0$ and values $x = \frac{\pi}{4} + \frac{\pi}{2}k$ ($k \in Z$).

If $A = 1$, then it will be equal for any values x. Therefore $y_{max} = \sqrt{2(1 + A)}$.

3. Find at what values x, the expression $y = \frac{5x^6 + 343}{x^5}$ takes the smallest value.

Solution: let's write the expression in the form $y = 5x + \frac{343}{x^5}$. In this case, we will not benefit from using $\frac{a+b}{2} \ge \sqrt{ab}$ ($a \ge 0$, $b \ge 0$):

$$\sqrt{5x \cdot \frac{343}{x^5}} = \sqrt{\frac{5 \cdot 343}{x^4}},$$

The variable x is under the root. If $x + x + x + x + x + \frac{343}{x^5}$, then

$$y_{min} = 6 \cdot \frac{x + x + x + x + x + \frac{343}{x^5}}{6} \ge 6\sqrt[6]{x^5 \cdot \frac{343}{x^5}} = 6\sqrt[6]{343}.$$

This means that the function $y(x)$ reaches the lowest value at $x = x = x = x = x = \frac{343}{x^5}$ and $x = \sqrt[6]{343}$.

4. Find the largest value of the function $y = \left(64 - x^6\right) \cdot x^6$.

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Solution: by the expression of the function, we find out that at $|x| \geq \sqrt[6]{64} = 2$ $y \leq 0$, and at $|x| < 2$ $y \geq 0$. Therefore , it is sufficient to consider the case $|x| < 2$. $|x| < 2$ that $64 - x^6 > 0$ and $x^6 \geq 0$. By inequality $ab \leq \left(\frac{a+b}{2}\right)^2$:

$$y_{max} = \left(\frac{\left(64 - x^6\right) + x^6}{2}\right)^2 = 32^2 = 1024.$$

This value is reached when $64 - x^6 = x^6$ and $x = \pm\sqrt[6]{32}$.

5. Find the smallest value of the function $y = \frac{x^6 + 15}{x}$ ($x > 0$).

Solution: Write the expression in the form $y = x^5 + \frac{15}{x}$. If applicable for two numbers $\frac{a+b}{2} \geq \sqrt{ab}$, then it leads to $y_{min} = 2\sqrt{x^5 \cdot \frac{15}{x}} = 2\sqrt{x^4 \cdot 15}$ This value is not constant because the variable is under the root. To get rid of this, we write in the form

$$y = x^5 + \frac{3}{x} + \frac{3}{x} + \frac{3}{x} + \frac{3}{x} + \frac{3}{x}$$

Then

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} \geq \sqrt[6]{a_1 \cdot a_2 \cdot \ldots \cdot a_6}, a_i > 0$$

From here $y_{\text{энг кичик}} = 6\sqrt[6]{x^5 \cdot \frac{243}{x^5}} = 6\sqrt[6]{243}$. This value is reached when $x^5 = \frac{3}{x} = \frac{3}{x} = \frac{3}{x} = \frac{3}{x} = \frac{3}{x}$ and $x = \sqrt[6]{3}$.

Thus, it can be seen that the knowledge obtained in the 9th grade of secondary school, long before studying the basics of differentiation, is quite enough to solve optimization problems of a certain level. It is considered advisable to devote more time to studying optimization problems in the school curriculum, as they contribute to the development of analytical and logical thinking of students, and the development of an internal desire to solve problems not only correctly, but also most quickly and efficiently. As our great teachers noted: "Mathematics is the basis of all exact sciences. A child who knows mathematics well will grow up prudent, will be able to work successfully in any field." These words should serve as a support for the education of worthy minds of our country.

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